Indian Statistical Institute, Bangalore
M. Math Second Year
Second Semester - Ergodic Theory

Midterm Exam — Date: February 22, 2019
Maximum marks: 30 — Duration: 2.30 hours

Answer any five, each question carries 6 marks

1. (a) Let $(X_i, \mathcal{B}_i, m_i)$ be two probability spaces and $T: X_1 \to X_2$ be a map. Suppose $\mathcal{S}_2$ be a semi-algebra generating $\mathcal{B}_2$ and for any $A \in \mathcal{S}_2$, $T^{-1}(A) \in \mathcal{B}_1$ and $m_1(T^{-1}(A)) = m_2(A)$. Prove that T is a mpt *(Marks:3)*.

   (b) Let (X, d) be a compact metric space and μ be a Borel probability measure on X. Let T be a mpt on X. Prove that $\liminf d(T^n x, x) = 0$ for a.e. $x \in X$.

2. Let T be a mpt of a probability space $(X, \mathcal{B}, m)$. Prove that T is ergodic if and only if any measurable function f with $f(T(x)) = f(x)$ a.e. is constant a.e.

3. (a) Let T be a ergodic mpt on a probability space X. Prove that any measurable $f: X \to \mathbb{R}$ such that $f(Tx) \geq f(x)$ is constant a.e.

   (b) Determine all probability measures on $X = [0, 1]$ so that $T: X \to X$ given by $T(x) = x^2$ is a ergodic mpt *(Marks: 3)*.

4. Define two-sided $(p_0, p_1, \cdots, p_{k-1})$-shift and show it is ergodic.

5. (a) Let T be a mpt of a probability space $(X, \mathcal{B}, m)$. Prove that T is ergodic if and only if for any $A, B \in \mathcal{B}$, $\frac{1}{n}\sum_{k=0}^{n-1} m(T^{-k}A \cap B) \to m(A)m(B)$.

   (b) Let T be a ergodic mpt of a probability space $(X, \mathcal{B}, m)$ and $f \geq 0$ be measurable. If $\int f dm = \infty$, prove that $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \to \infty$ a.e. (*Marks: 3*).

6. (a) Prove that finite union of sets of density zero is of density zero (*marks: 2*).

   (b) A mpt T on (X, μ) is weakly mixing if and only if for any ergodic mpt S on (Y, λ), the product system $T \times S$ on $(X \times Y, \mu \otimes \lambda)$ is also ergodic.

7. (a) Prove that $T: S^2 \to S^2$ given by $T(x, y) = (xy, xe^{i\frac{\pi}{4}})$ is strong mixing.

   (b) Let X be a metric space and T be a isometric transformation. If μ is a probability measure on X such that T is weakly mixing on (X, μ), prove that $\mu = \delta_x$ for some $x \in X$ (*Marks: 4*).